

C^1 -Classification of gapped parent Hamiltonians of quantum spin chains with local symmetry

YOSHIKO OGATA*

Graduate School of Mathematical Sciences
The University of Tokyo, Komaba, Tokyo, 153-8914, Japan

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Abstract

We consider the family of gapped Hamiltonians introduced in [FNW, N] on the quantum spin chain $\bigotimes_{\mathbb{Z}} M_n$, *with local symmetry* given by a group G . The G -symmetric gapped Hamiltonians are given by triples (k, u, V) , where u is a projective unitary representation of G on a finite dimensional space \mathbb{C}^k , and V is an isometry from \mathbb{C}^k to $\mathbb{C}^n \otimes \mathbb{C}^k$. We show that Hamiltonians H_0, H_1 , given by the triples $(k^{(0)}, u^{(0)}, V^{(0)})$ and $(k^{(1)}, u^{(1)}, V^{(1)})$ are C^1 -equivalent if the projective representations $u^{(0)}$ and $u^{(1)}$ are unitary equivalent.

1 Introduction

Recently, gapped ground state phases attract a lot of attentions [CGW1][CGW2][SPC][BMNS][CGLW][BO][SW]. In quantum spin systems, we say two gapped Hamiltonians are equivalent if and only if they are connected by a continuous path of uniformly gapped Hamiltonians. When we further require the path to be C^1 , we call it C^1 -classification. It is known that the ground state structure is an invariant of the C^1 -classification [BMNS]. The ultimate goal should be classifying all the gapped Hamiltonians in the world. When we impose symmetry, the classification problem raises different mathematical question [SPC][CGLW]. Two Hamiltonians which are equivalent in the classification without symmetry may be no longer in the same class if we consider the classification *with* symmetry.

The general framework of classification with symmetry was considered in [BN]. However, it is in general a hard problem to guarantee the existence of the spectral gap along the path rigorously. As a result, examples of gapped Hamiltonians are quite limited in quantum spin systems whose spatial dimensions are larger than one. However, for one dimensional systems, there is a recipe of gapped Hamiltonians [FNW]. We completely classified Hamiltonians given by this recipe, without breaking translation invariance in [BO]. In [FNW], the existence of the spectral gap is guaranteed by the primitivity of the associated completely positive map. We carried out the classification, by showing that the space of primitive maps, with an upper bound on Kraus rank, is connected, in [BO]. Recently, an alternative proof of this fact was introduced in [SW]. Both proof [BO], and [SW] have their own advantages. The proof in [SW] is simpler and we don't need to care about the detailed structure, i.e., we don't need to know *how the primitivity is guaranteed* in details. Because of this simplicity, this proof can provide the analyticity of the path. On the other hand, we find in [O1] and [O2] that the argument in [BO] can be extended to the non-primitive maps. It also has an advantage in constructing examples, as we know how the primitivity is guaranteed concretely.

In this paper, we consider the classification of the class of gapped Hamiltonians, given by the recipe of [FNW], *with* symmetry. The G -symmetric gapped Hamiltonians are given by triples

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(k, u, V) , where u is a projective unitary representation of G on a finite dimensional space \mathbb{C}^k , and V is an isometry from \mathbb{C}^k to $\mathbb{C}^n \otimes \mathbb{C}^k$. This problem is considered in [SPC]. The difference here is that we do not break translation invariance. Mathematically, our task is to guarantee the primitivity of the associated CP-maps, *and* the symmetry. As the recipe in [FNW] cares only about the primitive maps, we can use the method in [SW]. About this problem, [SW] is much stronger method than the one in [BO].

Now let us state our result in detail. See Appendix A for notations. For $\mathbb{N} \ni n \geq 2$, let \mathcal{A} be the finite dimensional C^* -algebra $\mathcal{A} = M_n$, the algebra of $n \times n$ matrices. Throughout this article, this n is fixed as the dimension of the spin under consideration, and we fix an orthonormal basis $\{\psi_\mu\}_{\mu=1}^n$ of \mathbb{C}^n . We denote the set of all finite subsets in \mathbb{Z} by \mathfrak{S} . The number of elements in a finite set $\Lambda \subset \mathbb{Z}$ is denoted by $|\Lambda|$. When we talk about intervals in \mathbb{Z} , $[a, b]$ for $a \leq b$, means the interval in \mathbb{Z} , i.e., $[a, b] \cap \mathbb{Z}$. We denote the set of all finite intervals by \mathfrak{I} . For each $x \in \mathbb{Z}$, we let $\mathcal{A}_{\{x\}}$ be an isomorphic copy of \mathcal{A} and for any finite subset $\Lambda \subset \mathbb{Z}$, $\mathcal{A}_\Lambda = \otimes_{x \in \Lambda} \mathcal{A}_{\{x\}}$ is the local algebra of observables. For finite Λ , the algebra \mathcal{A}_Λ can be regarded as the set of all bounded operators acting on a Hilbert space $\otimes_{x \in \Lambda} \mathbb{C}^n$. We use this identification freely. If $\Lambda_1 \subset \Lambda_2$, the algebra \mathcal{A}_{Λ_1} is naturally embedded in \mathcal{A}_{Λ_2} by tensoring its elements with the identity. Finally, the algebra $\mathcal{A}_{\mathbb{Z}}$ is given as the inductive limit of the algebras \mathcal{A}_Λ with $\Lambda \in \mathfrak{S}$. We denote the set of local observables by $\mathcal{A}_{\mathbb{Z}}^{\text{loc}} = \bigcup_{\Lambda \in \mathfrak{S}} \mathcal{A}_\Lambda$. For any $x \in \mathbb{Z}$, let τ_x be the shift operator by x on $\mathcal{A}_{\mathbb{Z}}$.

The local symmetry is introduced as follows. Let G be a group. Let $U : G \rightarrow \mathcal{U}_n$ be a unitary representation of G on \mathbb{C}^n . Here, \mathcal{U}_n denotes the set of all unitary matrices on \mathbb{C}^n . By β_U , we denote the product action of G on $\mathcal{A}_{\mathbb{Z}}$ induced by U , i.e.,

$$\beta_{U_g}(A) = (\cdots \otimes U_g \otimes U_g \otimes U_g \otimes \cdots) A (\cdots \otimes U_g^{-1} \otimes U_g^{-1} \otimes U_g^{-1} \otimes \cdots), \quad (1)$$

for any $A \in \mathcal{A}_{\mathbb{Z}}$ and $g \in G$.

An interaction is a map Φ from \mathfrak{S} into $\mathcal{A}_{\mathbb{Z}}^{\text{loc}}$ such that $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X) = \Phi(X)^*$ for $X \in \mathfrak{S}$. An interaction Φ is translation invariant if $\Phi(X + j) = \tau_j(\Phi(X))$, for all $j \in \mathbb{Z}$ and $X \in \mathfrak{S}$. Furthermore, it is of finite range if there exists an $m \in \mathbb{N}$ such that $\Phi(X) = 0$, for X with diameter larger than m . In this case, we say that the interaction length of Φ is less than or equal to m . We denote the set of all translation invariant finite range interactions by \mathcal{J} . Furthermore, the set of all translation invariant interactions with interaction length less than or equal to m is denoted by \mathcal{J}_m . For the product action β_U of a group G , we say an interaction Φ is β_U -invariant if $\beta_g(\Phi(X)) = \Phi(X)$ for all $X \in \mathfrak{S}$. A natural number $m \in \mathbb{N}$ and an element $h \in \mathcal{A}_{[0, m-1]}$, define an interaction Φ_h by

$$\Phi_h(X) := \begin{cases} \tau_x(h), & \text{if } X = [x, x + m - 1] \text{ for some } x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

for $X \in \mathfrak{S}$. If h is β_U -invariant, i.e., $\beta_{U_g}(h) = h$ for all $g \in G$, then the interaction Φ_h is also β_U -invariant. A Hamiltonian associated with Φ is a net of self-adjoint operators $H_\Phi := ((H_\Phi)_\Lambda)_{\Lambda \in \mathfrak{I}}$ such that

$$(H_\Phi)_\Lambda := \sum_{X \subset \Lambda} \Phi(X). \quad (3)$$

Note that $(H_\Phi)_\Lambda \in \mathcal{A}_\Lambda$. Let us specify what we mean by gapped Hamiltonian in this paper.

Definition 1 A Hamiltonian $H := (H_\Lambda)_{\Lambda \in \mathfrak{I}}$ associated with a positive translation invariant finite range interaction is gapped if there exists $\gamma > 0$ and $N_0 \in \mathbb{N}$ such that the difference between the smallest and the next-smallest eigenvalue of H_Λ , is bounded below by γ , for all finite intervals $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$.

In this definition, the smallest eigenvalue can be degenerated in general.

Now we introduce the C^1 -classification of gapped Hamiltonians with β_U -symmetry. We say $\Phi : [0, 1] \ni t \mapsto \Phi(t) \in \mathcal{J}$ is a continuous and piecewise C^1 -path if for each $X \in \mathfrak{S}$, the path $[0, 1] \ni t \mapsto \Phi(t; X) \in \mathcal{A}_X$ is continuous and piecewise C^1 with respect to the norm topology.

Definition 2 (C^1 -classification of gapped Hamiltonians with symmetry) Let $U : G \rightarrow \mathcal{U}_n$ be a unitary representation of a group G on \mathbb{C}^n . Let H_0, H_1 be gapped Hamiltonians associated with β_U -invariant interactions $\Phi_0, \Phi_1 \in \mathcal{J}$. We say that H_0, H_1 are C^1 -equivalent with β_U -symmetry if the following conditions are satisfied.

1. There exists $m \in \mathbb{N}$ and a continuous and piecewise C^1 -path $\Phi : [0, 1] \rightarrow \mathcal{J}_m$ such that $\Phi(0) = \Phi_0, \Phi(1) = \Phi_1$.
2. Let $H(t)$ be the Hamiltonian associated with $\Phi(t)$ for each $t \in [0, 1]$. There are $\gamma > 0, N_0 \in \mathbb{N}$, and finite intervals $I(t) = [a(t), b(t)]$, whose endpoints $a(t), b(t)$ smoothly depending on $t \in [0, 1]$, such that for all finite intervals $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$, the smallest eigenvalue of $H(t)_\Lambda$ is in $I(t)$ and the rest of the spectrum is in $[b(t) + \gamma, \infty)$.
3. For each $t \in [0, 1]$, $\Phi(t)$ is β_U -invariant.

Remark 3 We write $H_0 \simeq_{C^1, U} H_1$ when H_0, H_1 are C^1 -equivalent with β_U -symmetry. If furthermore the path $\Phi(t)$ can be taken in \mathcal{J}_m we write $H_0 \simeq_{C^1, U, m} H_1$.

Now let us introduce the family of Hamiltonians we consider in this paper. As an input, we prepare the following triple.

Definition 4 Let G be a group and $c : G \times G \rightarrow \mathbb{T}$ a 2-cocycle of G . Let $U : G \rightarrow \mathcal{U}_n$ be a unitary representation of G on \mathbb{C}^n . Let k be a natural number, u_g a projective unitary representation of G on \mathbb{C}^k with respect to the 2-cocycle c , and V an isometry from \mathbb{C}^k to $\mathbb{C}^n \otimes \mathbb{C}^k$. We denote by $\mathcal{SP}(n, G, U, c)$ the set of all such triple (k, u, V) which satisfies the following conditions.

(i) For any $g \in G$, we have

$$(U_g \otimes u_g) V = V u_g. \quad (4)$$

(ii) Define $\mathbf{v} = \mathbf{v}_{(k, u, V)} = (v_1, \dots, v_n) \in \mathbb{M}_k^{\times n}$ by

$$Vx = \sum_{\mu=1}^n \psi_\mu \otimes v_\mu^* x, \quad x \in \mathbb{C}^k. \quad (5)$$

(Recall that $\{\psi_\mu\}_\mu$ is the fixed CONS of \mathbb{C}^n .) Define the completely positive map $T_{\mathbf{v}} : \mathbb{M}_k \rightarrow \mathbb{M}_k$ by

$$T_{\mathbf{v}}(X) := \sum_{\mu=1}^n v_\mu X v_\mu^*, \quad X \in \mathbb{M}_k. \quad (6)$$

Then $T_{\mathbf{v}}$ is primitive.

The definition of primitivity can be found in [W], for example.

From the n -tuple of $k \times k$ matrices $\mathbf{v} = \mathbf{v}_{(k, u, V)} = (v_1, \dots, v_n) \in \mathbb{M}_k^{\times n}$ associated with $(k, u, V) \in \mathcal{SP}(n, G, U, c)$, we construct our interaction following the recipe of [FNW]. For $l \in \mathbb{N}$ and $\mu^{(l)} = (\mu_0, \mu_1, \dots, \mu_{l-1}) \in \{1, \dots, n\}^{\times l}$, we use the notation

$$\widehat{v_{\mu^{(l)}}} := v_{\mu_0} v_{\mu_1} \cdots v_{\mu_{l-1}} \in \mathbb{M}_k, \quad \widehat{\psi_{\mu^{(l)}}} := \bigotimes_{i=0}^{l-1} \psi_{\mu_i} \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n. \quad (7)$$

Definition 5 Let $(k, u, V) \in \mathcal{SP}(n, G, U, c)$ and $\mathbf{v} = \mathbf{v}_{(k, u, V)} = (v_1, \dots, v_n) \in \mathbf{M}_k^{\times n}$ the n -tuple associated to it. For each $l \in \mathbb{N}$, define $\Gamma_{l, \mathbf{v}} : \mathbf{M}_k \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ by

$$\Gamma_{l, \mathbf{v}}(X) = \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \left(\text{Tr } X \left(\widehat{v_{\mu^{(l)}}} \right)^* \right) \widehat{\psi_{\mu^{(l)}}}, \quad X \in \mathbf{M}_k, \quad (8)$$

and set $\mathcal{G}_{l, \mathbf{v}} := \text{Ran } \Gamma_{l, \mathbf{v}} \subset \bigotimes_{i=0}^{l-1} \mathbb{C}^n$. Furthermore, we denote by $G_{l, \mathbf{v}}$ the orthogonal projection onto $\mathcal{G}_{l, \mathbf{v}}$ in $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$. We set $h_{l, \mathbf{v}} := 1 - G_{l, \mathbf{v}}$ and $\Phi_{l, \mathbf{v}} := \Phi_{h_{l, \mathbf{v}}}$. (Recall (2).)

It is known that this recipe gives a Hamiltonian with a spectral gap [FNW].

Lemma 6 Let $(k, u, V) \in \mathcal{SP}(n, G, U, c)$ and $\mathbf{v} = \mathbf{v}_{(k, u, V)} = (v_1, \dots, v_n) \in \mathbf{M}_k^{\times n}$ the n -tuple associated to it. Then, for any $m \in \mathbb{N}$, the interaction $\Phi_{m, \mathbf{v}}$ is β_U -invariant. For $m \geq k^4 + 1$ the Hamiltonian $H_{\Phi_{m, \mathbf{v}}}$ is gapped.

Proof. To show the first statement, let us consider a positive operator

$$X_m := \sum_{\mu^{(m)}, \nu^{(m)} \in \{1, \dots, n\}^{\times m}} \left(\text{Tr} \left(\left(\widehat{v_{\mu^{(m)}}} \right)^* \widehat{v_{\nu^{(m)}}} \right) \right) \left| \widehat{\psi_{\mu^{(m)}}} \right\rangle \left\langle \widehat{\psi_{\nu^{(m)}}} \right|.$$

We claim that the support $s(X_m)$ of X_m is $G_{m, \mathbf{v}}$. To see $G_{m, \mathbf{v}} \leq s(X_m)$, let us consider an arbitrary $\xi \in \bigotimes_{i=0}^{m-1} \mathbb{C}^n$, and set $Z_\xi := \sum_{\mu^{(m)}} \left\langle \xi, \widehat{\psi_{\mu^{(m)}}} \right\rangle \left(\widehat{v_{\mu^{(m)}}} \right)^* \in \mathbf{M}_k$. For any $Y \in \mathbf{M}_k$, we have $\langle \xi, \Gamma_{m, \mathbf{v}}(Y) \rangle = \text{Tr } Y Z_\xi$. We also have $\langle \xi, X \xi \rangle = \text{Tr } Z_\xi Z_\xi^*$. Therefore, from the Cauchy-Schwartz inequality, we obtain

$$|\langle \xi, \Gamma_{m, \mathbf{v}}(Y) \rangle|^2 = |\text{Tr } Y Z_\xi|^2 \leq \text{Tr } Y Y^* \cdot \text{Tr } Z_\xi Z_\xi^* = \text{Tr } Y Y^* \cdot \langle \xi, X \xi \rangle,$$

for any $Y \in \mathbf{M}_k$ and $\xi \in \bigotimes_{i=0}^{m-1} \mathbb{C}^n$. This implies $G_{m, \mathbf{v}} \leq s(X_m)$. To prove $G_{m, \mathbf{v}} \geq s(X_m)$, let $\{e_{i, j}^{(k)}\}_{i, j=1}^k$ be the matrix units of \mathbf{M}_k . By the straightforward calculation, we obtain

$$X_m = \sum_{i, j=1}^k \left| \Gamma_{m, \mathbf{v}} \left(e_{i, j}^{(k)} \right) \right\rangle \left\langle \Gamma_{m, \mathbf{v}} \left(e_{i, j}^{(k)} \right) \right|.$$

This proves $G_{m, \mathbf{v}} \geq s(X_m)$.

From this, we see that in order to show $G_{m, \mathbf{v}}$ is β_U -invariant, it suffices to show that X_m is β_U -invariant. For this, we define a completely positive map $\mathbb{E} : \mathbf{M}_n \otimes \mathbf{M}_n \rightarrow \mathbf{M}_k$ by

$$\mathbb{E}(A \otimes X) := V^* (A \otimes X) V = \sum_{\mu, \nu=1}^n \langle \psi_\mu, A \psi_\nu \rangle v_\mu X v_\nu^* \quad A \in \mathbf{M}_n, \quad X \in \mathbf{M}_k.$$

For each $A \in \mathbf{M}_n$, we set $\mathbb{E}_A : \mathbf{M}_k \rightarrow \mathbf{M}_k$ by $\mathbb{E}_A(X) = V^* (A \otimes X) V$, $X \in \mathbf{M}_k$. From (i) of the Definition 4, we have

$$\begin{aligned} \mathbb{E}_{\text{Ad}(U_g^*)(A)}(X) &= V^* (U_g^* A U_g \otimes X) V = V^* (U_g \otimes u_g)^* (A \otimes u_g X u_g^*) (U_g \otimes u_g) V \\ &= (V u_g)^* (A \otimes u_g X u_g^*) V u_g = \text{Ad } u_g^* \circ \mathbb{E}_A \circ \text{Ad } u_g(X). \end{aligned} \quad (9)$$

(Here $\text{Ad}(U_g^*)(X) := U_g^* X U_g$ etc.) On the other hand, by the straightforward calculation we obtain the following formula for any $A_0, \dots, A_{m-1} \in \mathbf{M}_n$,

$$\text{Tr}_{\bigotimes_{i=0}^{m-1} \mathbf{M}_n} \left(X_m \bigotimes_{i=0}^{m-1} A_i \right) = \text{Tr}_{\mathbf{M}_k} (\mathbb{E}_{A_0} \circ \dots \circ \mathbb{E}_{A_{m-1}}(1)). \quad (10)$$

Now we show the β_U -invariance of X_m . For all $A_0, \dots, A_{m-1} \in M_n$,

$$\begin{aligned} \text{Tr}_{\bigotimes_{i=0}^{m-1} M_n} \left(\text{Ad} \left(U_g^{\bigotimes_{i=0}^{m-1}} \right) (X_m) \bigotimes_{i=0}^{m-1} A_i \right) &= \text{Tr}_{\bigotimes_{i=0}^{m-1} M_n} \left(X_m \left(\bigotimes_{i=0}^{m-1} \text{Ad} (U_g^*) (A_i) \right) \right) \\ &= \text{Tr}_{M_k} \left(\mathbb{E}_{\text{Ad } U_g^* (A_0)} \circ \dots \circ \mathbb{E}_{\text{Ad } U_g^* (A_{m-1})} (1) \right) = \text{Tr}_{M_k} \left(\mathbb{E}_{A_0} \circ \dots \circ \mathbb{E}_{A_{m-1}} (1) \right) = \text{Tr}_{\bigotimes_{i=0}^{m-1} M_n} \left(X_m \bigotimes_{i=0}^{m-1} A_i \right). \end{aligned}$$

Here, in the second and the fourth equality, we used (10), and for the third equality, we used (9). This proves $\beta_{U_g}(X_m) = X_m$, for all $g \in G$.

The second statement follows from [FNW], except for the condition $m \geq k^4 + 1$ which is discussed in Lemma 3.1 of [BO] and basically quantum Wielandt's inequality [SPWC]. \square

Definition 7 Let G be a group and $c : G \times G \rightarrow \mathbb{T}$ a 2-cocycle of G . Let (u_0, \mathbb{C}^{k_0}) , (u_1, \mathbb{C}^{k_1}) be finite dimensional projective unitary representations of G with respect to the 2-cocycle c . We say $(u^{(0)}, \mathbb{C}^{k_0})$ and $(u^{(1)}, \mathbb{C}^{k_1})$ are unitary equivalent if $k := k_0 = k_1$ and there exists a unitary matrix $W \in \mathcal{U}_k$ such that $W u_g^{(0)} = u_g^{(1)} W$, for all $g \in G$.

Here is the main Theorem of this paper.

Theorem 8 Let G be a group and $c : G \times G \rightarrow \mathbb{T}$ a 2-cocycle of G . Let $U : G \rightarrow \mathcal{U}_n$ be a unitary representation of G on \mathbb{C}^n . Let $(k^{(0)}, u^{(0)}, V^{(0)})$ and $(k^{(1)}, u^{(1)}, V^{(1)})$ be elements in $SP(n, G, U, c)$, and $\mathbf{v}_0 := \mathbf{v}(k^{(0)}, u^{(0)}, V^{(0)})$, $\mathbf{v}_1 := \mathbf{v}(k^{(1)}, u^{(1)}, V^{(1)})$, the n -tuples associated to them via (5). Assume that the projective representations $(u^{(0)}, \mathbb{C}^{k_0})$ and $(u^{(1)}, \mathbb{C}^{k_1})$ are unitary equivalent. Then we have $H_{\Phi_{\mathbf{v}_0}, \mathbf{v}_0} \simeq_{C^1, U, m} H_{\Phi_{\mathbf{v}_1}, \mathbf{v}_1}$, for any $m \geq k^4 + 1$.

2 Proof of Theorem 8

We introduce an equivalence relation in $SP(n, G, U, c)$.

Definition 9 Let $(k^{(0)}, u^{(0)}, V^{(0)})$ and $(k^{(1)}, u^{(1)}, V^{(1)})$ be elements in $SP(n, G, U, c)$. We say $(k^{(0)}, u^{(0)}, V^{(0)})$ and $(k^{(1)}, u^{(1)}, V^{(1)})$ are equivalent, if the following holds.

1. $k := k^{(0)} = k^{(1)}$.
2. There exists a map $u : [0, 1] \times G \rightarrow \mathcal{U}_k$ such that for any $g \in G$, $[0, 1] \ni t \rightarrow u(\cdot, g) \in \mathcal{U}_k$ is C^∞ .
3. There exists a C^∞ -map $V : [0, 1] \rightarrow B(\mathbb{C}^k, \mathbb{C}^n \otimes \mathbb{C}^k)$ such that $(k, u(t, \cdot), V(t)) \in SP(n, G, U, c)$, for all $t \in [0, 1]$, with $(k, u(0, \cdot), V(0)) = (k^{(0)}, u^{(0)}, V^{(0)})$ and $(k, u(1, \cdot), V(1)) = (k^{(1)}, u^{(1)}, V^{(1)})$.

When $(k^{(0)}, u^{(0)}, V^{(0)})$ and $(k^{(1)}, u^{(1)}, V^{(1)})$ are equivalent, we write $(k^{(0)}, u^{(0)}, V^{(0)}) \simeq_{SP} (k^{(1)}, u^{(1)}, V^{(1)})$.

In order to prove Theorem 8, we show $(k^{(0)}, u^{(0)}, V^{(0)}) \simeq_{SP} (k^{(1)}, u^{(1)}, V^{(1)})$. Note that if u is a projective representation of G with respect to a 2-cocycle c , for any $W \in \mathcal{U}_k$, the map $G \ni g \mapsto W u_g W^*$ defines a projective unitary representation of G with respect to c . We denote it by $\text{Ad } W(u)$.

Lemma 10 Let $(k, u, V) \in SP(n, G, U, c)$ and $W \in \mathcal{U}_k$. Then the triple $(k, \text{Ad } W(u), (\mathbb{I} \otimes W) V W^*)$ belongs to $SP(n, G, U, c)$. In particular, we have

$$(k, u, V) \simeq_{SP} (k, \text{Ad } W(u), (\mathbb{I} \otimes W) V W^*).$$

Proof. It is clear that $\text{Ad } W(u)$ is a projective unitary representation of G with respect to a 2-cocycle c , and that $(\mathbb{I} \otimes W) V W^*$ is an isometry. It is straightforward to check

$$(U_g \otimes \text{Ad } W(u_g)) (\mathbb{I} \otimes W) V W^* = (\mathbb{I} \otimes W) V W^* \text{Ad } W(u_g).$$

Let v be the n -tuple associated to (k, u, V) via (5). We then have

$$(\mathbb{I} \otimes W) V W^* x = \sum_{\mu=1}^n \psi_\mu \otimes (W v_\mu W)^* x, \quad x \in \mathbb{C}^k.$$

The primitivity of T_v implies the primitivity of $T_{(W v_\mu W^*)_{\mu=1}^n}$. Therefore, we have $(k, \text{Ad } W(u), (\mathbb{I} \otimes W) V W^*) \in SP(n, G, U, c)$.

To show the second statement, let $H \in M_k$ be a self adjoint element such that $W = e^{iH}$. Then, the path $[0, 1] \ni t \mapsto \hat{W}(t) := e^{itH} \in U_k$ is C^∞ , and the path of the triple $[0, 1] \ni t \mapsto (k, \text{Ad } \hat{W}(t)(u), (\mathbb{I} \otimes \hat{W}(t)) V \hat{W}(t)^*)$ satisfies the conditions in Definition 9. To prove the third condition in Definition 9, we use the first statement of this Lemma, which we have just proven. \square

From Lemma 10, we see that in order to prove Theorem 8, it suffices to consider the case $k^{(0)} = k^{(1)}$ and $u^{(0)} = u^{(1)}$. Therefore, we prove the following Lemma.

Lemma 11 *Let (u, \mathbb{C}^k) be finite dimensional projective unitary representation of G with respect to the 2-cocycle c . Let $V_0, V_1 : \mathbb{C}^k \rightarrow \mathbb{C}^n \otimes \mathbb{C}^k$ be isometries such that $(k, u, V_0), (k, u, V_1) \in SP(n, G, U, c)$. Then we have*

$$(k, u, V_0) \simeq_{SP} (k, u, V_1).$$

In order to prove this, we first investigate the structure of the isometry V . We consider all the equivalence classes of finite dimensional irreducible unitary c -projective representations of G . Let $\{(\pi_\alpha, V_\alpha)\}_\alpha$ be the set of representatives of them. Then we obtain the irreducible decompositions of u_g and $U_g \otimes u_g$ [K]. Note that the proof of Schur's Lemma and the irreducible decomposition for usual representations works for unitary projective representations. For each irreducible c -projective representation π_α , there exist numbers $m_\alpha, m'_\alpha \in \mathbb{N} \cup \{0\}$. There also exist unitaries $W : \mathbb{C}^k \rightarrow \bigoplus_{\alpha: m_\alpha \neq 0} V_\alpha \otimes \mathbb{C}^{m_\alpha}$, $W' : \mathbb{C}^n \otimes \mathbb{C}^k \rightarrow \bigoplus_{\alpha: m'_\alpha \neq 0} V_\alpha \otimes \mathbb{C}^{m'_\alpha}$ and we have

$$W u_g W^* = \bigoplus_{\alpha: m_\alpha \neq 0} \pi_\alpha(g) \otimes \mathbb{I}_{m_\alpha}, \quad (11)$$

$$W' (U_g \otimes u_g) W'^* = \bigoplus_{\alpha: m'_\alpha \neq 0} \pi_\alpha(g) \otimes \mathbb{I}_{m'_\alpha}. \quad (12)$$

We have the following Lemma.

Lemma 12 *Let $(k, u, V) \in SP(n, G, U, c)$. We consider the irreducible decompositions given in (11) and (12). Then $m_\alpha = 0$ if $m'_\alpha = 0$. Furthermore, if $m_\alpha \neq 0$, there exists an isometry $\omega_\alpha : \mathbb{C}^{m_\alpha} \rightarrow \mathbb{C}^{m'_\alpha}$ and they satisfy*

$$V = W'^* \left(\bigoplus_{\alpha: m_\alpha \neq 0} \mathbb{I}_{V_\alpha} \otimes \omega_\alpha \right) W.$$

Proof. For $m \in \mathbb{N}$, let $\{\chi_i^{(m)}\}_{i=1}^m$ be the standard basis of \mathbb{C}^m . Each element ξ in $\bigoplus_{\alpha: m_\alpha \neq 0} V_\alpha \otimes \mathbb{C}^{m_\alpha}$ can be decomposed as

$$\xi = \bigoplus_{\alpha: m_\alpha \neq 0} \sum_{i=1}^{m_\alpha} \xi_{\alpha, i} \otimes \chi_i^{(m_\alpha)},$$

with some $\xi_{\alpha,i} \in V_\alpha$. For α with $m_\alpha \neq 0$ and $i = 1, \dots, m_\alpha$, let us consider the subspace of $\bigoplus_{\beta: m_\beta \neq 0} V_\beta \otimes \mathbb{C}^{m_\beta}$ which consists of ξ such that $\xi_{\beta,j} = 0$ if $(\beta, j) \neq (\alpha, i)$. We denote the orthogonal projection onto this subspace by $P_{\alpha,i}$. Similarly, for α with $m'_\alpha \neq 0$ and $i = 1, \dots, m'_\alpha$ we define an orthogonal projection $P'_{\alpha,i}$ on $\bigoplus_{\alpha: m'_\alpha \neq 0} V_\alpha \otimes \mathbb{C}^{m'_\alpha}$. Substituting the decompositions (11) and (12) to (4), we obtain

$$\left(\bigoplus_{\alpha: m'_\alpha \neq 0} \pi_\alpha(g) \otimes \mathbb{I}_{m'_\alpha} \right) W' V W^* = W' V W^* \left(\bigoplus_{\alpha: m_\alpha \neq 0} \pi_\alpha(g) \otimes \mathbb{I}_{m_\alpha} \right), \quad g \in G. \quad (13)$$

Multiplying $P_{\alpha,i}$ from the left and $P_{\beta,j}$ from the right of this equation, we obtain

$$\pi_\alpha(g) P_{\alpha,i} W' V W^* P_{\beta,j} = P_{\alpha,i} W' V W^* P_{\beta,j} \pi_\beta(g), \quad g \in G,$$

for all α, β such that $m'_\alpha \neq 0$ and $m_\beta \neq 0$. Regarding $P_{\alpha,i} W' V W^* P_{\beta,j}$ a linear map from V_β to V_α , we apply Schur's Lemma. We then obtain $P_{\alpha,i} W' V W^* P_{\beta,j} = 0$ if $\alpha \neq \beta$. If $m_\alpha, m'_\alpha \neq 0$, there exists a scalar $C_{\alpha,i,j} \in \mathbb{C}$ such that $P_{\alpha,i} W' V W^* P_{\alpha,j} = C_{\alpha,i,j} \mathbb{I}_{V_\alpha}$. Set $\omega_\alpha := \sum_{\substack{1 \leq i \leq m_\alpha \\ 1 \leq j \leq m'_\alpha}} C_{\alpha,i,j} \left| \chi_i^{(m'_\alpha)} \right\rangle \left\langle \chi_j^{(m_\alpha)} \right| \in B(\mathbb{C}^{m_\alpha}, \mathbb{C}^{m'_\alpha})$ in this case. We then obtain

$$W' V W^* = \sum_{\substack{\alpha: \beta: m_\alpha \neq 0, m'_\beta \neq 0 \\ 1 \leq i \leq m_\alpha, 1 \leq j \leq m'_\beta}} P_{\alpha,i} W' V W^* P_{\beta,j} = \sum_{\substack{\alpha: m_\alpha \neq 0, m'_\alpha \neq 0 \\ 1 \leq i \leq m_\alpha, 1 \leq j \leq m'_\alpha}} P_{\alpha,i} W' V W^* P_{\alpha,j} = \bigoplus_{\alpha: m_\alpha \neq 0, m'_\alpha \neq 0} \mathbb{I}_{V_\alpha} \otimes \omega_\alpha.$$

As V is an isometry, we obtain

$$\bigoplus_{\alpha: m_\alpha \neq 0} \mathbb{I}_{V_\alpha} \otimes \mathbb{C}^{m_\alpha} = \mathbb{I} = (W' V W^*)^* W' V W^* = \bigoplus_{\alpha: m_\alpha \neq 0, m'_\alpha \neq 0} \mathbb{I}_{V_\alpha} \otimes \omega_\alpha^* \omega_\alpha.$$

If there exists an α such that $m_\alpha \neq 0$ and $m'_\alpha = 0$, this equality can not hold. Therefore, we have $m'_\alpha \neq 0$ if $m_\alpha \neq 0$. Furthermore, $\omega_\alpha : \mathbb{C}^{m_\alpha} \rightarrow \mathbb{C}^{m'_\alpha}$ is an isometry. \square

Proof of Lemma 11. For U and u , we consider the irreducible decompositions (11), (12). Applying Lemma 12 to V_i , $i = 0, 1$, we obtain isometries $\omega_{i,\alpha} : \mathbb{C}^{m_\alpha} \rightarrow \mathbb{C}^{m'_\alpha}$, such that

$$V_i = W'^* \left(\bigoplus_{\alpha: m_\alpha \neq 0} \mathbb{I}_{V_\alpha} \otimes \omega_{i,\alpha} \right) W.$$

For each α with $m_\alpha \neq 0$, there exists a unitary $S_\alpha \in \mathcal{U}_{m'_\alpha}$ such that $S_\alpha \omega_{0,\alpha} = \omega_{1,\alpha}$. This is because $\omega_{0,\alpha}, \omega_{1,\alpha}$ are isometries. Let $H_\alpha \in M_{m'_\alpha}$ be a self-adjoint matrix such that $S_\alpha = e^{iH_\alpha}$. We set $H_\alpha = 0$ if $m'_\alpha \neq 0$ and $m_\alpha = 0$. We define $H := \bigoplus_{\alpha: m'_\alpha \neq 0} \mathbb{I}_{V_\alpha} \otimes H_\alpha$.

We would like to connect (k, u, V_0) and (k, u, V_1) by a smooth path in $SP(n, G, U, c)$, by connecting \mathbb{I} and e^{iH} suitably. In order for that, we recall the necessary and sufficient condition for the primitivity, introduced in [SW]. See Appendix B. We use the notations in Appendix B. We define a $B(\mathbb{C}^k, \mathbb{C}^n \otimes \mathbb{C}^k)$ -valued entire analytic function $V(z)$ by

$$V(z) := W'^* e^{izH} \left(\bigoplus_{\alpha: m_\alpha \neq 0} \mathbb{I}_{V_\alpha} \otimes \omega_{0,\alpha} \right) W, \quad z \in \mathbb{C}. \quad (14)$$

Note that $V(0) = V_0$ and $V(1) = V_1$. From the definition (14), and the decompositions (11), (12), we obtain

$$(U_g \otimes u_g) V(z) = V(z) u_g, \quad z \in \mathbb{C}, \quad g \in G. \quad (15)$$

As in (5), we define $v_\mu(z)$

$$V(z)x = \sum_{\mu=1}^n \psi_\mu \otimes (v_\mu(\bar{z}))^* x, \quad x \in \mathbb{C}^k, \quad z \in \mathbb{C}.$$

Note that $\mathbb{C} \ni z \mapsto v_\mu(z)$ is entire analytic, for each $\mu = 1, \dots, n$. We write $\mathbf{v}(z) := (v_1(z), \dots, v_n(z))$. By the same calculation as in (9), we obtain

$$\text{Ad } u_g^* \circ T_{\mathbf{v}(z)} \circ \text{Ad } u_g = T_{\mathbf{v}(z)}, \quad z \in \mathbb{C}, \quad g \in G. \quad (16)$$

For each $\boldsymbol{\mu} := (\mu_i^{(k^4)})_{i=1}^{k^2}$, where $\mu_i^{(k^4)} \in \{1, \dots, n\}^{\times k^4}$, $i = 1, \dots, k^2$, we define an entire analytic function

$$f_{\boldsymbol{\mu}}(z) := \left\langle \zeta, \bigotimes_{i=1}^{k^2} \left(\widehat{v_{\mu_i^{(k^4)}}(z)} \otimes \mathbb{I} \right) \Omega \right\rangle.$$

We set

$$\mathcal{Z} := \bigcap_{\boldsymbol{\mu}} \{z \in \mathbb{C} \mid f_{\boldsymbol{\mu}}(z) = 0\}.$$

Here the intersection is taken over all $\boldsymbol{\mu} := (\mu_i^{(k^4)})_{i=1}^{k^2}$, where $\mu_i^{(k^4)} \in \{1, \dots, n\}^{\times k^4}$, $i = 1, \dots, k^2$. From Lemma 13, $T_{\mathbf{v}(z)}$ is primitive if and only if there exists $\boldsymbol{\mu}$ such that $f_{\boldsymbol{\mu}}(z) \neq 0$. Therefore, $T_{\mathbf{v}(z)}$ is primitive if and only if $z \notin \mathcal{Z}$. In particular, we have $0, 1 \notin \mathcal{Z}$. This means at least one of $f_{\boldsymbol{\mu}}$ is not identically zero. As each $f_{\boldsymbol{\mu}}(z)$ is entire analytic and at least one of them is not identically zero, the intersection of \mathcal{Z} and a ball $\{z \in \mathbb{C} \mid |z| \leq 2\}$ is a finite set.

Let $\tilde{\varpi} : [0, 1] \rightarrow \mathbb{C}$ be a path in \mathbb{C} given by $\tilde{\varpi}(t) = t$, $t \in [0, 1]$. As $\mathcal{Z} \cap \{z \in \mathbb{C} \mid |z| \leq 2\}$ is a finite set, we can deform this $\tilde{\varpi}$ and obtain a C^∞ -path $\varpi : [0, 1] \rightarrow \mathbb{C}$ with $\varpi(0) = 0$, $\varpi(1) = 1$ such that $\varpi(t) \notin \mathcal{Z}$ for all $t \in [0, 1]$. In particular, $T_{\mathbf{v}(\varpi(t))}$ is primitive for $t \in [0, 1]$. From this primitivity, the spectral radius $r_{T_{\mathbf{v}(\varpi(t))}}$ of $T_{\mathbf{v}(\varpi(t))}$ is strictly positive, and it is a non-degenerate eigenvalue of $T_{\mathbf{v}(\varpi(t))}$. Let $P_{\{r_{T_{\mathbf{v}(\varpi(t))}}\}}^{T_{\mathbf{v}(\varpi(t))}}$ be the spectral projection of $T_{\mathbf{v}(\varpi(t))}$ onto $\{r_{T_{\mathbf{v}(\varpi(t))}}\}$. Then $e_{\mathbf{v}(\varpi(t))} := P_{\{r_{T_{\mathbf{v}(\varpi(t))}}\}}^{T_{\mathbf{v}(\varpi(t))}}(\mathbb{I}_k)$ is a strictly positive element in M_k and there exists a faithful state $\varphi_{\mathbf{v}(\varpi(t))}$ on M_k such that $P_{\{r_{T_{\mathbf{v}(\varpi(t))}}\}}^{T_{\mathbf{v}(\varpi(t))}}(X) = \varphi_{\mathbf{v}(\varpi(t))}(X) e_{\mathbf{v}(\varpi(t))}$, for $X \in M_k$. By (16), $e_{\mathbf{v}(\varpi(t))}$ and u_g commute for all $t \in [0, 1]$ and $g \in G$. Note that $r_{T_{\mathbf{v}(\varpi(0))}} = r_{T_{\mathbf{v}(\varpi(1))}} = 1$ and $e_{\mathbf{v}(\varpi(0))} = e_{\mathbf{v}(\varpi(1))} = 1$. Furthermore, the maps $[0, 1] \ni t \mapsto r_{T_{\mathbf{v}(\varpi(t))}}, e_{\mathbf{v}(\varpi(t))}, \varphi_{\mathbf{v}(\varpi(t))}$ are C^∞ . Hence, setting

$$\hat{v}_\mu(t) := r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} v_\mu(\varpi(t)) e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}}, \quad t \in [0, 1], \quad \mu = 1, \dots, n,$$

we obtain C^∞ maps in M_k . We set $\hat{\mathbf{v}}(t) := (\hat{v}_1(t), \dots, \hat{v}_n(t))$, $t \in [0, 1]$. Note that $\hat{\mathbf{v}}(0) = \mathbf{v}(\varpi(0)) = \mathbf{v}(0)$ and $\hat{\mathbf{v}}(1) = \mathbf{v}(\varpi(1)) = \mathbf{v}(1)$. By this definition, $T_{\hat{\mathbf{v}}(t)}$ is a unital completely positive map. As $T_{\mathbf{v}(\varpi(t))}$ is primitive, this $T_{\hat{\mathbf{v}}(t)}$ is also primitive.

We define a C^∞ -path $\hat{V} : [0, 1] \rightarrow B(\mathbb{C}^k, \mathbb{C}^n \otimes \mathbb{C}^k)$, by

$$\hat{V}(t)x := \sum_{\mu=1}^n \psi_\mu \otimes \hat{v}_\mu(t)^* x, \quad x \in \mathbb{C}^k, \quad t \in [0, 1].$$

Note that $\hat{V}(0) = V(0) = V_0$, $\hat{V}(1) = V(1) = V_1$. In order to prove Lemma 11, it suffices to show $(k, u, \hat{V}(t)) \in \mathcal{SP}(n, G, U, c)$, for all $t \in [0, 1]$. Since $T_{\hat{\mathbf{v}}(t)}$ is unital, $\hat{V}(t)$ is an isometry. As we already know that $T_{\hat{\mathbf{v}}(t)}$ is primitive, what we have to check is (i) of Definition 4. This can be

checked as follows. For any $x \in \mathbb{C}^k$,

$$\begin{aligned}
(U_g \otimes u_g) \hat{V}(t)x &= (U_g \otimes u_g) \left(\sum_{\mu=1}^n \psi_\mu \otimes \hat{v}_\mu(t)^* x \right) = r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} \left(\sum_{\mu=1}^n U_g \psi_\mu \otimes u_g e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} (v_\mu(\varpi(t)))^* e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} x \right) \\
&= r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} \left(\sum_{\mu=1}^n U_g \psi_\mu \otimes e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} u_g (v_\mu(\varpi(t)))^* e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} x \right) \\
&= r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} \left(\mathbb{I}_n \otimes e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} \right) (U_g \otimes u_g) \left(\sum_{\mu=1}^n \psi_\mu \otimes (v_\mu(\varpi(t)))^* e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} x \right) \\
&= r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} \left(\mathbb{I}_n \otimes e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} \right) (U_g \otimes u_g) V \left(\overline{\varpi(t)} \right) e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} x = r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} \left(\mathbb{I}_n \otimes e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} \right) V \left(\overline{\varpi(t)} \right) u_g e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} x \\
&= r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} \left(\mathbb{I}_n \otimes e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} \right) V \left(\overline{\varpi(t)} \right) e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} u_g x = r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} \left(\mathbb{I}_n \otimes e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} \right) \left(\sum_{\mu=1}^n \psi_\mu \otimes (v_\mu(\varpi(t)))^* e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} u_g x \right) \\
&= \left(\sum_{\mu=1}^n \psi_\mu \otimes r_{T_{\mathbf{v}(\varpi(t))}}^{-\frac{1}{2}} e_{\mathbf{v}(\varpi(t))}^{\frac{1}{2}} (v_\mu(\varpi(t)))^* e_{\mathbf{v}(\varpi(t))}^{-\frac{1}{2}} u_g x \right) = \sum_{\mu=1}^n \psi_\mu \otimes \hat{v}_\mu(t)^* u_g x = \hat{V}(t) u_g x, \quad g \in G, \quad t \in [0, 1].
\end{aligned}$$

In the third and the seventh equality, we used the commutativity of $e_{\mathbf{v}(\varpi(t))}$ and u_g . In the sixth equality, we used (15). This completes the proof. \square

Proof of Theorem 8. As $(u^{(0)}, \mathbb{C}^{k^{(0)}})$ and $(u^{(1)}, \mathbb{C}^{k^{(1)}})$ are unitary equivalent, we have $k := k^{(0)} = k^{(1)}$ and there exists a unitary matrix $W \in \mathcal{U}_k$ such that $W u_g^{(0)} = u_g^{(1)} W$, $g \in G$. By Lemma 10, we have $(k^{(0)}, u^{(0)}, V^{(0)}) \simeq_{SP} (k^{(0)}, \text{Ad } W(u^{(0)}), (\mathbb{I} \otimes W) V^{(0)} W^*) = (k, u^{(1)}, (\mathbb{I} \otimes W) V^{(0)} W^*)$. Furthermore, by Lemma 11, we obtain $(k, u^{(1)}, (\mathbb{I} \otimes W) V^{(0)} W^*) \simeq_{SP} (k, u^{(1)}, V_1) = (k^{(1)}, u^{(1)}, V_1)$. Applying Lemma 6, the same argument as [BO] implies Theorem 8. \square

A Notations

For $k \in \mathbb{N}$, the set of all $k \times k$ matrices over \mathbb{C} is denoted by \mathbf{M}_k . Furthermore, we denote the set of unitary elements of \mathbf{M}_k by \mathcal{U}_k . For $A \in \mathbf{M}_k$, we denote the map $\mathbf{M}_k \ni X \mapsto AXA^* \in \mathbf{M}_k$ by $\text{Ad } A$. For a linear map Γ , $\text{Ker } \Gamma$, and $\text{Ran } \Gamma$ denote the kernel and the range of Γ respectively. For a finite dimensional Hilbert space, bracket \langle, \rangle denotes the inner product of the space under consideration. We denote the set of all bounded linear maps from Hilbert space \mathcal{H} to \mathcal{K} is denoted by $B(\mathcal{H}, \mathcal{K})$. For a subspace \mathfrak{K} , \mathfrak{K}^\perp means the orthogonal complement of \mathcal{H} .

B Primitivity

In this section we recall the necessary and sufficient condition for $T_{\mathbf{v}}$, (given for an n -tuple of $k \times k$ matrices $\mathbf{v} \in \mathbf{M}_k^{\times n}$), to be primitive. This condition was introduced in [SW]. See [SW] for the detail. Let $\{\chi_i^{(k)}\}_{i=1}^k$ (resp. $\{\chi_i^{(k^2)}\}_{i=1}^{k^2}$) be a standard basis of \mathbb{C}^k (resp. \mathbb{C}^{k^2}). We set $\Omega := \sum_{i=1}^k \chi_i^{(k)} \otimes \chi_i^{(k)}$. We also set $\zeta := \sum_{\sigma \in S_{k^2}} \text{sgn } \sigma \chi_{\sigma(1)}^{(k^2)} \otimes \chi_{\sigma(2)}^{(k^2)} \otimes \cdots \otimes \chi_{\sigma(k^2)}^{(k^2)}$. Here, S_{k^2} is the symmetric group of degree k^2 and $\text{sgn } \sigma$ is the signature of $\sigma \in S_{k^2}$.

Lemma 13 ([SW]) *The completely positive map T_v is primitive if and only if there exists $\mu := (\mu_i^{(k^4)})_{i=1}^{k^2}$, where $\mu_i^{(k^4)} \in \{1, \dots, n\}^{\times k^4}$, $i = 1, \dots, k^2$ such that*

$$\left\langle \zeta, \bigotimes_{i=1}^{k^2} \left(v_{\mu_i^{(k^4)}} \otimes \mathbb{I} \right) \Omega \right\rangle \neq 0.$$

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